

# Wildly primitive extensions

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**Abstract.** A finite separable extension of a field is called primitive if there are no intermediate extensions. The most interesting primitive extensions of a local field with finite residue field are the wildly ramified ones, and our aim here is to parametrise them in a canonical manner.

*Meinem Doktorvater gewidmet*

## 1. Introduction

(1) Let  $p$  be a prime number and let  $K$  be a  $p$ -field, namely a local field with finite residue field of characteristic  $p$ . All extensions of  $K$  appearing below are assumed to be *separable* over  $K$ . A finite extension  $E$  of  $K$  is called *primitive* if  $[E : K] > 1$  and if the only extensions of  $K$  in  $E$  are  $K$  and  $E$ . It is easy to see that a tamely ramified extension of  $K$  is primitive if and only if it is either unramified of prime degree or totally ramified of prime degree  $l \neq p$ ; the latter are parametrised by sections of the projection  $K^\times/K^{\times l} \rightarrow \mathbf{Z}/l\mathbf{Z}$  coming from the valuation on  $K$ . Thus tamely ramified primitive extensions are easy to classify.

We are interested in primitive  $p$ -extensions of the  $p$ -field  $K$  — those whose degree is a power of  $p$ . If a primitive  $p$ -extension  $E$  is ramified, then clearly it is wildly and totally ramified over  $K$ ; we say that  $E$  is *wildly primitive* for short. The only primitive  $p$ -extension of  $K$  which is not wildly primitive is the unramified extension of degree  $p$  over  $K$ .

(2) Extensions of degree  $p$  over  $K$  are always primitive (and wildly so if ramified). They have been parametrised by Del Corso and Dvornicich [7] if  $K$  has characteristic 0 and by the present author [2] in general. More recently, Del Corso, Dvornicich and Monge [8] have studied wildly primitive extensions of  $K$  of degree  $p^n$  when  $\text{char } K = 0$ ; see also Pati [16]

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when moreover  $n$  is prime. Our aim here is to parametrise all primitive  $p$ -extensions of an arbitrary  $p$ -field  $K$  by generalising one of the main results of [2] (the case  $n = 1$ ) and its proof, and to compute their discriminants. For some historical remarks, see §9.

## 2. Notations and results

(3) Let  $k$  be the residue field of  $K$  and  $q = \text{Card } k$ . For every  $n > 0$ , put  $e_n = p^n - 1$ ,  $K_n = K(\sqrt[n]{1})$  (where  $\sqrt[n]{1}$  stands for an element of order  $e_n$  in the multiplicative group), and  $L_n = K_n(\sqrt[n]{K_n^\times})$ , so that  $K_n$  is the unramified extension of  $K$  of degree equal to the order  $s_n$  of  $\bar{q} \in (\mathbf{Z}/e_n\mathbf{Z})^\times$ , and  $L_n$  is the maximal abelian extension of  $K_n$  of exponent dividing  $e_n$ . The ramification index (resp. the residual degree) of  $L_n$  over  $K$  is  $e_n$  (resp.  $s_n e_n$ ).

Note that  $L_n$  is a tamely ramified galoisian extension of  $K$  of group  $G_n = \text{Gal}(L_n|K)$ ; it is split over  $K$  in the sense that the short exact sequence  $\{1\} \rightarrow T \rightarrow G_n \rightarrow G_n/T \rightarrow \{1\}$  has a section, where  $T$  is the inertia subgroup of  $G_n$ . If  $K$  has characteristic 0, then the  $p$ -torsion subgroup  ${}_p L_n^\times$  of  $L_n^\times$  has order  $p$  (because  $L_1$  contains  $\sqrt[p]{-p}$ ).

View  $\overline{L_n^\times} = L_n^\times / L_n^{\times p}$  (resp.  $\overline{L_n^+} = L_n^+ / \wp(L_n^+)$ , where  $\wp(x) = x^p - x$ ) as an  $\mathbf{F}_p[G_n]$ -module if  $K$  has characteristic 0 (resp.  $p$ ). Our first main result is the following parametrisation of the set of primitive  $p$ -extensions of  $K$  of fixed degree :

(4) *The set of  $K$ -isomorphism classes of primitive extensions  $E$  of  $K$  of degree  $p^n$  is in canonical bijection with the set of simple submodules  $D$  of the  $\mathbf{F}_p[G_n]$ -module  $L_n^\times / L_n^{\times p}$ , resp.  $L_n^+ / \wp(L_n^+)$ , of degree  $n$ , under the correspondence  $EL_n = L_n(\sqrt[p]{D})$ , resp.  $EL_n = L_n(\wp^{-1}(D))$ .*

In characteristic 0, this is a more precise version of the main result of [8, 3.2], as we specify the extension  $L_n$  explicitly. The proof (§6) is a generalisation from the case  $n = 1$  treated in [2]. As there, we also determine the structure of the filtered  $\mathbf{F}_p[G_n]$ -module  $L_n^\times / L_n^{\times p}$ , resp.  $L_n^+ / \wp(L_n^+)$ , in §7.

Later we shall define the *level* of a simple submodule  $D$  of  $\overline{L_n^\times}$  or of  $\overline{L_n^+}$  in terms of the natural filtration on these  $G_n$ -modules. Our second main result relates the level of  $D$  to the differential exponent of the corresponding primitive  $p$ -extension  $E$  of  $K$  (33). This allows us to compute — in principle — the number of primitive extensions  $E$  of  $K$  of degree  $p^n$  with a given differential exponent.

This parametrisation is illustrated in §8 in the simplest cases of primitive quartic or octic extensions of dyadic fields ( $p = 2$ ,  $n = 2$  or  $3$ ).

(5) *Remarks.* Let  $\tilde{K}$  be a maximal galoisian extension of  $K$  containing the extensions  $L_n$  (3), and let  $M_n$  be the maximal abelian extension of  $L_n$  in  $\tilde{K}$  of exponent  $p$ , so that the direct limit  $V = \varinjlim L_n$  is the maximal tamely ramified extension of  $K$  in  $\tilde{K}$  and the direct limit  $B = \varinjlim M_n$  is the maximal abelian extension of  $V$  in  $\tilde{K}$  of exponent  $p$ , namely  $B = V(\sqrt[p]{V^\times})$ , resp.  $B = V(\wp^{-1}(V))$ . Let  $W$  be the compositum of all wildly primitive extensions  $E$  of  $K$  in  $\tilde{K}$ , or equivalently the compositum in  $\tilde{K}$  of their galoisian closures  $\hat{E}$  over  $K$ . It follows from (4) that if  $[E : K] = p^n$ , then  $\hat{E} \subset M_n$ , and hence  $W \subset B$ . It is likely that  $W = B$ , just as the compositum of all degree- $p$  extensions of  $K$  in  $\tilde{K}$  is  $M_1$  [2, Proposition 33].

(6) Let  $\Gamma = \text{Gal}(V|K)$ . The structure of the  $\mathbf{F}_p[[\Gamma]]$ -module  $\text{Gal}(B|V)$  has been determined in [5, 39], resp. [5, 42], by showing that the dual  $\mathbf{F}_p[[\Gamma]]$ -module  $\overline{V^\times} = V^\times/V^{\times p}$ , resp.  $\overline{V^+} = V^+/\wp(V^+)$ , is isomorphic to  $\mathbf{F}_p[[\Gamma]]^{[K:\mathbf{Q}_p]} \oplus \mathbf{F}_p$ , resp.  $\mathbf{F}_p[[\Gamma]]^{(\mathbf{N})}$ .

### 3. Solvable primitive $l$ -extensions

(7) Let us recall from [3] a general algebraic result which we need. Fix a field  $F$  and a maximal galoisian extension  $\tilde{F}$  of  $F$ . All extensions of  $F$  appearing below are assumed to be contained in  $\tilde{F}$ . A finite extension  $E$  of  $F$  is called *solvable* if the group  $G = \text{Gal}(\hat{E}|K)$  is solvable, where  $\hat{E}$  is the galoisian closure of  $E$  over  $F$ . Galois proved that if  $E$  is a solvable primitive (1) extension of  $F$ , then  $[E : K] = l^n$  for some prime  $l$  and some  $n > 0$ .

(8) Also fix the prime  $l$  and the integer  $n > 0$ . Let  $N$  be a minimal normal subgroup of  $G$ . As  $[E : F] = l^n$ , the  $\mathbf{F}_l$ -dimension  $N$  is  $n$ . The group  $\text{Gal}(\tilde{F}|F)$  acts on  $N$  through its quotient  $G/N$ ; the resulting  $\mathbf{F}_l$ -representation  $\rho$  of  $\text{Gal}(\tilde{F}|F)$  is irreducible and its image is solvable. The extension  $E$  of  $F$  is uniquely determined (up to  $F$ -isomorphism) by the pair consisting of  $\rho$  and the extension  $\hat{E}$  of  $L$ , where  $F \subset L \subset \hat{E}$  is such that  $N = \text{Gal}(\hat{E}|L)$ . More precisely,

(9) *Let  $F$  be a field,  $l$  a prime number, and  $n > 0$  an integer. The map sending a primitive solvable extension  $E$  of  $F$  of degree  $l^n$  to its galoisian closure  $\hat{E}$  over  $F$  establishes a bijection between the set of  $F$ -isomorphism classes of such  $E$  with the set of pairs  $(\rho, M)$  consisting of an irreducible degree- $n$   $\mathbf{F}_l$ -representation  $\rho$  of  $\text{Gal}(\tilde{F}|F)$  with solvable image and an abelian extension  $M$  of exponent  $l$  and degree  $l^n$  of the fixed field  $L_\rho = \tilde{F}^{\text{Ker}(\rho)}$  of  $\rho$  such that  $M$  is galoisian over  $F$  and the resulting conjugation action of  $\text{Gal}(L_\rho|F)$  on  $\text{Gal}(M|L_\rho)$  is given by  $\rho$ .*  $\square$

(10) In particular, given a primitive  $l$ -extension  $E$  of degree  $l^n$  over  $F$ , there is a finite galoisian extension  $L$  of  $F$ , uniquely determined by  $E$ , such that  $\hat{E} = EL$ ,  $\text{Gal}(\hat{E}|L)$  is an  $\mathbf{F}_l$ -space of dimension  $n$ , and the  $\text{Gal}(L|F)$ -module  $\text{Gal}(\hat{E}|L)$  is faithful and simple.

In what follows, we will take  $F$  to be our  $p$ -field  $K$  and  $l$  to be the prime  $p$ .

#### 4. Sections and conjugates

(11) Another purely algebraic ingredient we need is a lemma used in [8, 3.3], where its proof is attributed to [1, 6.1]. For the convenience of the reader, we briefly reproduce it here.

(12) *Let  $l$  be a prime number. Let  $G$  be a finite group which has a normal subgroup  $A$  of order prime to  $l$  and index a power of  $l$ , and let  $C$  be a simple  $\mathbf{F}_l[G]$ -module of finite degree  $> 1$ . Then  $H^1(G, C) = \{0\}$  and  $H^2(G, C) = \{0\}$ .*

*Proof.* The inflation-restriction sequence in this situation is the exact sequence [17, Chapitre VII, Proposition 4]

$$\{0\} \rightarrow H^1(G/A, C^A) \xrightarrow{\text{Inf}} H^1(G, C) \xrightarrow{\text{Res}} H^1(A, C).$$

Since the orders of  $A$  and  $C$  are relatively prime, we have  $H^1(A, C) = \{0\}$ . The same reason, together with the hypotheses that  $G/A$  is an  $l$ -group and that  $\dim_{\mathbf{F}_l} C > 1$ , implies that  $C^A = \{0\}$  and hence  $H^1(G/A, C^A) = \{0\}$ . Therefore  $H^1(G, C) = \{0\}$ . This being so, the sequence

$$\{0\} \rightarrow H^2(G/A, C^A) \xrightarrow{\text{Inf}} H^2(G, C) \xrightarrow{\text{Res}} H^2(A, C).$$

is exact [17, Chapitre VII, Proposition 5], and a similar argument leads to the conclusion  $H^2(G, C) = \{0\}$ .  $\square$

(13) As the group  $H^2(G, C)$  classifies extensions of  $G$  by the  $G$ -module  $C$ , and as  $H^1(G, C)$  classifies sections of the neutral extension of  $G$  by  $C$  up to conjugation, we get the following equivalent statement (in the multiplicative notation) : every extension  $\{1\} \rightarrow C \rightarrow \Gamma \rightarrow G \rightarrow \{1\}$  of  $G$  by  $C$  admits a section  $G \rightarrow \Gamma$ , and any two sections are conjugate by an element of  $\Gamma$ .

#### 5. Irreducible $\mathbf{F}_p$ -representations of $\text{Gal}(\tilde{K}|K)$

(14) Let's return to our local field  $K$  and recall that our aim is to parametrise the set of primitive  $p$ -extensions of  $K$ . The general algebraic

result of §3 leads us to classify irreducible degree- $n$   $\mathbf{F}_p$ -representations  $\rho$  of  $\text{Gal}(\tilde{K}|K)$ , where  $\tilde{K}$  is a maximal galoisian extension of  $K$ . Note that every finite extension of  $K$  is (separable by hypothesis (1) and) solvable in the sense of (7), and the condition that the image of  $\rho$  be solvable is automatically satisfied. In this context, recall one of the main results from [4], employing the notation introduced in §2.

(15) *Every irreducible  $\mathbf{F}_p$ -representation of  $\text{Gal}(\tilde{K}|K)$  of degree  $n$  factors through the quotient  $G_n = \text{Gal}(L_n|K)$ .*  $\square$

## 6. The proof of the parametrisation

(16) Before entering into the details, let us outline the strategy of the proof of Theorem (4). We will first show that for every primitive extension  $E$  of degree  $p^n$  over  $K$ ,

- a) *the extension  $EL_n$  of  $L_n$  is abelian of exponent  $p$  and degree  $p^n$ ,*
- b) *the extension  $EL_n$  of  $K$  is a galoisian, and*
- c) *the resulting  $\mathbf{F}_p[G_n]$ -module  $C = \text{Gal}(EL_n|L_n)$  is simple.*

Recall (3) that the  $p$ -torsion subgroup  ${}_pL_n^\times$  of  $L_n^\times$  has order  $p$  if  $K$  has characteristic 0. Therefore, once we establish a), there will be a unique  $n$ -dimensional subspace  $D$  of  $L_n^\times/L_n^{\times p}$ , resp. of  $L_n^+/\wp(L_n^+)$ , such that  $EL_n = L_n(\sqrt[p]{D})$  if  $\text{char } K = 0$  and  $EL_n = L_n(\wp^{-1}(D))$  if  $\text{char } K = p$ . Once we establish b), we will know that  $D$  is  $G_n$ -stable, and, once we establish c), we will know that the  $\mathbf{F}_p[G_n]$ -module  $D$  is simple, because there are canonical isomorphisms  $D = \text{Hom}(C, {}_pL_n^\times)$ , resp.  $D = \text{Hom}(C, \mathbf{F}_p)$ .

We will then show that conversely,

d) *for every simple  $\mathbf{F}_p[G_n]$ -submodule  $D$  of  $L_n^\times/L_n^{\times p}$  or of  $L_n^+/\wp(L_n^+)$ , of degree  $n$ , there is a primitive extension  $E$  of  $K$  of degree  $p^n$ , unique up to  $K$ -isomorphism, such that  $EL_n = L_n(\sqrt[p]{D})$  or  $EL_n = L_n(\wp^{-1}(D))$  respectively.*

(17) Let's prove (4). Let  $E$  be a primitive extension of  $K$  of degree  $p^n$ . By the general algebraic theory of §3 (applicable because  $E$  is solvable in the sense of (7)),  $E$  determines a finite galoisian extension  $L$  of  $K$  such that the galoisian closure  $\hat{E}$  of  $E$  over  $K$  is  $\hat{E} = EL$ , the group  $\text{Gal}(\hat{E}|L)$  is an  $\mathbf{F}_p$ -space of dimension  $n$ , and it is faithful and simple as a  $\text{Gal}(L|K)$ -module. By (15), we have  $L \subset L_n$ ; in particular,  $L$  is tamely ramified over  $K$ . We claim that the extensions  $\hat{E}$  and  $L_n$  of  $L$  are *linearly disjoint*.

This is clear if  $E$  is the unramified degree- $p$  extension of  $K$  (in which case  $L = K$ ,  $[\hat{E} : K] = p$ , and  $[L_1 : K] = (p-1)^2$ ). Otherwise,  $\hat{E}$  is totally ramified of degree  $p^n$  over  $L$  whereas  $L_n$  is tamely ramified over  $L$ , and

the claim follows. Therefore  $EL_n = \hat{E}L_n$  has the properties *a)*, *b)* and *c)* of (16).

(18) To establish the claim *d)* of (16), take a simple submodule  $D$  of  $L_n^\times/L_n^{\times p}$  or of  $L_n^+/\wp(L_n^+)$  (according as  $K$  has characteristic 0 or  $p$ ), of dimension  $n$  over  $\mathbf{F}_p$ . Put  $M = L_n(\sqrt[p]{D})$  or  $M = L_n(\wp^{-1}(D))$  respectively and put  $C = \text{Gal}(M|L_n)$ . Note that  $M$  is galoisian over  $K$ , and that the  $\mathbf{F}_p[G_n]$ -module  $C$  is simple because  $C = \text{Hom}(D, {}_pL_n^\times)$  or  $C = \text{Hom}(D, \mathbf{F}_p)$  respectively.

(19) If  $n = 1$ , so that the order of  $C$  is  $p$  and the order of  $G_1$  is  $(p-1)^2$ , we have  $H^2(G_1, C) = \{0\}$  and  $H^1(G_1, C) = \{0\}$ . Therefore the extension  $\text{Gal}(M|K)$  of  $G_1$  by  $C$  splits, and any two sections are conjugate. In other words, there is a degree- $p$  extension  $E$  of  $K$ , unique up to  $K$ -isomorphism, such that  $EL_1 = M$ , and we are done.

(20) There is a similar argument when  $n > 1$ . Consider the maximal unramified extension  $p$ -extension  $P$  of  $K$  in  $L_n$ . The subgroup  $\text{Gal}(L_n|P)$  of  $G_n$  is obviously invariant under conjugation, of order prime to  $p$ , and of index a power of  $p$ . We can therefore apply (13) to our situation and conclude that the extension  $\text{Gal}(M|K)$  of  $G_n$  by  $C$  splits, and that any two sections are conjugate. This means that there is a degree- $p^n$  extension  $E$  of  $K$ , unique up to  $K$ -isomorphism, such that  $EL_n = M$ .

(21) It remains to show that  $E$  is primitive. Indeed, suppose there is an intermediate extension  $K \subset F \subset E$ . Then  $FL_n$  is galoisian over  $K$  and  $\text{Gal}(M|FL_n)$  is a  $G_n$ -stable subspace of  $C$ , therefore either  $FL_n = L_n$  or  $FL_n = M$ , which implies that either  $F = K$  or  $F = E$ . This completes the proof of Theorem (4) following the strategy outlined in (16).  $\square\square\square$

(22) *Remarks.* Another strategy for proving (16)*d)* would be to consider  $L = L_n^{\text{Ker}(\rho)}$ , where  $\rho$  is the action of  $G_n$  on  $C$ , and to show directly that there is an abelian extension  $N$  of  $L$  of exponent  $p$  and degree  $p^n$ , unique up to  $L$ -isomorphism, which is galoisian over  $K$  and such that  $NL_n = M$ . If there is such an  $N$ , then (9) implies the existence, uniqueness, and primitivity of  $E$ .

(23) If  $E$  is a wildly primitive extension of  $K$  of degree  $p^n$ ,  $D$  its parameter,  $M = EL_n$  (so that  $M = L_n(\sqrt[p]{D})$  or  $M = L_n(\wp^{-1}(D))$  respectively), and  $\rho$  the action of  $G_n = \text{Gal}(L_n|K)$  on  $D$ , then the action of  $G_n$  on the  $\mathbf{F}_p$ -space  $C = \text{Gal}(M|L_n)$  is  $\rho^* \otimes \omega$ , where  $\rho^*$  is the contragradient (dual) of  $\rho$  and  $\omega : G_n \rightarrow \mathbf{F}_p^\times$  is the character giving the action of  $G_n$  on  ${}_pL_n^\times$  (resp. on  $\mathbf{F}_p$ ) if  $K$  has characteristic 0 (resp.  $p$ ), just as in the case  $n = 1$  [2, Lemma 15]. Note that the (wild) ramification subgroup  $\text{Gal}(M|K)_1$  of  $\text{Gal}(M|K)$  is  $\text{Gal}(M|K)_1 = C$ . It is also clear that

the group  $\text{Aut}_K(E)$  is trivial unless  $E$  is cyclic over  $K$  (of degree  $p$ ).

(24) We have parametrised primitive extensions  $E$  of  $K$  of degree  $p^n$  by simple submodules  $D$  of the  $\mathbf{F}_p[G_n]$ -module  $L_n^\times/L_n^{\times p}$  or  $L_n^+/\wp(L_n^+)$  of degree  $n$ . We could equally well parametrise them by their galoisian closures  $\hat{E}$  as in (9), since the above proof characterises the  $\hat{E}$  which arise. Indeed, a finite galoisian extension  $F$  of  $K$  is of the form  $\hat{E}$  for some primitive extension  $E$  of  $K$  of degree  $p^n$  if and only if,  $F'$  being the maximal tamely ramified extension of  $K$  in  $F$ , two properties hold :

i) the ramification subgroup  $H = \text{Gal}(F|F')$  of  $G = \text{Gal}(F|K)$  is an  $\mathbf{F}_p$ -space of dimension  $n$ , and

ii) the  $\mathbf{F}_p[G/H]$ -module  $H$  is faithful and simple.

Also, we need to look for  $F'$  only among the subextensions of  $L_n$ .

(25) Clearly, a primitive *galoisian* extension of any field is cyclic of prime degree. We claim that if  $E$  is a primitive extension of the  $p$ -field  $K$  whose *galoisian closure*  $\hat{E}$  is a *totally ramified  $p$ -extension* of  $K$ , then  $\hat{E} = E$  (and hence  $E$  is ramified cyclic of degree  $p$  over  $K$ ). Indeed, the hypothesis on  $\hat{E}$  implies that the maximal tamely ramified extension of  $K$  in  $\hat{E}$  is  $K$ . By the preceding remark,  $\text{Gal}(\hat{E}|K)$  is a faithful simple  $\mathbf{F}_p[\text{Gal}(K|K)]$ -module. It follows that  $\hat{E}|K$  is cyclic of degree  $p$ , and hence  $\hat{E} = E$ .

## 7. Little galoisian modules

(26) Our understanding of primitive extensions of  $K$  cannot be complete without working out the structure of the  $\mathbf{F}_p[G_n]$ -modules  $L_n^\times/L_n^{\times p}$  and  $L_n^+/\wp(L_n^+)$ , respectively when  $K$  has characteristic 0 and  $p$ . This was determined by Iwasawa [12] (see also [8, 4.4]) in characteristic 0 and in [5] in general. In this §, we recall these structure theorems and compute the discriminant of a wildly primitive extension  $E$  of degree  $p^n$  over  $K$  in terms of its *parameter*  $D$ , the simple  $\mathbf{F}_p[G_n]$ -submodule of  $\overline{L_n^\times}$  or  $\overline{L_n^+}$ , of degree  $n$ , such that  $EL_n = L_n(\sqrt[p]{D})$  or  $EL_n = L_n(\wp^{-1}(D))$ , associated to  $E$  by (4).

(27) Suppose that  $K$  is a finite extension of  $\mathbf{Q}_p$ . The  $\mathbf{F}_p[G_n]$ -module  $\overline{L_n^\times} = L_n^\times/L_n^{\times p}$  is isomorphic to  ${}_pL^\times \oplus \mathbf{F}_p[G_n]^{[K:\mathbf{Q}_p]} \oplus \mathbf{F}_p$ .  $\square$

(28) Suppose that the  $p$ -field  $K$  has characteristic  $p$ . The  $\mathbf{F}_p[G_n]$ -module  $\overline{L_n^+} = L_n^+/\wp(L_n^+)$  is isomorphic to  $\mathbf{F}_p \oplus \mathbf{F}_p[G_n]^{(\mathbf{N})}$ .  $\square$

(29) What is important in both cases is the natural filtration on the  $\mathbf{F}_p[G_n]$ -modules  $\overline{L_n^\times}$  or  $\overline{L_n^+}$  which was studied in detail in [5]. Consider the problem of computing the discriminant of  $E$  over  $K$  in terms of  $D$ . Put

$M = EL_n$  and  $C = \text{Gal}(M|L_n)$ . Note that the ramification filtration on  $C$  has a *unique ramification break*  $\gamma(C)$  because the  $G_n$ -module  $C$  is simple and the ramification filtration is  $G_n$ -stable.

We denote by  $\mathfrak{p}_{L_n}$  the unique maximal ideal of the ring of integers of  $L_n$ , and we put  $U_{L_n}^i = 1 + \mathfrak{p}_{L_n}^i$  for every  $i > 0$ .

(30) For defining the *level* of the simple submodule  $D$  of  $\overline{L_n^\times}$  or  $\overline{L_n^+}$  of degree  $n$ , suppose first that  $K$  has characteristic 0 and let  $e_{L_n}$  be the ramification index of  $L_n$  over  $\mathbf{Q}_p$ .

Notice that  $e_{L_n} = c_n \cdot (p - 1)$  for some integer  $c_n > 0$ . The filtration on  $\overline{L_n^\times}$  is given by the images  $\bar{U}_{L_n}^i$  of  $U_{L_n}^i$  for various  $i > 0$ . Put  $\bar{U}_{L_n}^0 = \overline{L_n^\times}$  by convention. As  $D$  is a *simple* submodule of  $\overline{L_n^\times}$ , there is a *unique*  $i \in \mathbf{N}$  such that  $D \subset \bar{U}_{L_n}^i$  but  $D \cap \bar{U}_{L_n}^{i+1} = \{1\}$ , because the filtration is  $G_n$ -stable. We define the *level* of  $D$  to be  $\delta(D) = pc_n - i$ .

We have  $\delta(D) \in [0, pc_n]$ , and if  $\delta(D) \equiv 0 \pmod{p}$ , then  $n = 1$  and either  $\delta(D) = 0$ ,  $D = \bar{U}_{L_1}^{pc_1}$  (which is  $G_1$ -isomorphic to  ${}_pL_1^\times$ ), and  $E$  is unramified of degree  $p$  over  $K$ , or  $\delta(D) = pc_1$ ; the latter lines  $D$  and the corresponding extensions  $M$  of  $L_1$  and  $E$  of  $K$  are said to be *très ramifiées*.

(31) Now suppose that  $K$  has characteristic  $p$ . The filtration on  $\overline{L_n^+}$  is given by the images  $\overline{\mathfrak{p}_{L_n}^i}$  ( $i \in \mathbf{Z}$ ). As before, and for the same reason, since  $D$  is a *simple* submodule of  $\overline{L_n^+}$ , there is a *unique*  $i \in \mathbf{Z}$  such that  $D \subset \overline{\mathfrak{p}_{L_n}^i}$  but  $D \cap \overline{\mathfrak{p}_{L_n}^{i+1}} = \{0\}$ . We define the *level* of  $D$  to be  $\delta(D) = -i$ .

We have  $\delta(D) \in \mathbf{N}$ , and if  $\delta(D) \equiv 0 \pmod{p}$ , then  $n = 1$ ,  $\delta(D) = 0$ ,  $D = \overline{\mathfrak{p}_{L_1}^0}$  (which is  $G_1$ -isomorphic to  $\mathbf{F}_p$ ), and  $E$  is unramified of degree  $p$  over  $K$ . There is no analogue of *très ramifiées* lines or extensions.

(32) For finite extensions  $L$  of  $K$  and  $M$  of  $L$ , denote the *differential exponent* (resp. ramification index) of  $M|L$  by  $d_{M|L}$  (resp.  $e_{M|L}$ ), and recall that  $d_{M|K} = d_{M|L} + d_{L|K}e_{M|L}$  [17, Chapitre III, Proposition 8]. If  $M|L$  is galoisian of group  $G$ , then  $d_{M|L} = \sum_{i \in \mathbf{N}} (g_i - 1)$ , where  $g_i$  is the order of the higher ramification subgroup  $G_i \subset G$  (in the lower numbering) [17, Chapitre IV, Proposition 4]. If  $L$  is tame over  $K$ , then  $d_{L|K} = e_{L|K} - 1$  [17, Chapitre III, Proposition 13]. So a good measure of the *wildness* of  $L$  over  $K$  in general is  $\varepsilon(L) = d_{L|K} - (e_{L|K} - 1)$ , and a good name for the invariant  $\varepsilon(L)$  would be the *differential excess* of  $L|K$ ; it was used by Serre in his mass formula for totally ramified extensions of  $K$  of fixed degree.

(33) Let  $E$  be a wildly primitive extension of  $K$ ,  $p^n = [E : K]$  its degree,  $\varepsilon(E)$  its differential excess (32), and  $\gamma(C)$  the unique ramification break (29) of  $C = \text{Gal}(M|L_n)$ , where  $M = EL_n$ . Let  $D$  be the simple



$\mathbf{F}_p[G_n]$ -submodule of  $\overline{L_n^\times}$  or  $\overline{L_n^+}$ , of degree  $n$ , such that  $M = L_n(\sqrt[p]{D})$  or  $M = L_n(\wp^{-1}(D))$ , and  $\delta(D)$  its level as in (30) or (31). We have  $\gamma(C) = \delta(D) = \varepsilon(E)$ .

*Proof.* See [2, 34] for the case  $n = 1$ ; the same proof works for  $n > 0$ . The equality  $\gamma(C) = \delta(D)$  follows from a certain orthogonality relation recalled there (where the level of  $D$  was defined to be  $-\delta(D)$ ). Apply (32) to get

$$\begin{array}{ccc} E & \xrightarrow{e_n} & M \\ p^n \uparrow & & \uparrow p^n \\ K & \xrightarrow{e_n} & L_n \end{array} \quad d_{E|K} \begin{array}{ccc} E & \xrightarrow{e_n-1} & M \\ \uparrow & & \uparrow (1+\gamma(C))(p^n-1) \\ K & \xrightarrow{e_n-1} & L_n, \end{array}$$

where the numbers along the arrows in the first (resp. second) square indicate ramification indices (resp. differential exponents) of the corresponding extension. Compute  $d_{M|K}$  along the two paths from  $K$  to  $M$  and compare to get

$$(1 + \gamma(C))(p^n - 1) + (e_n - 1)p^n = (e_n - 1) + d_{E|K}e_n,$$

and recall that  $e_n = p^n - 1$ , to conclude that  $\gamma(C) = \varepsilon(E)$ .  $\square$

## 8. Some quartic and octic examples

(34) Taking  $p = 2$  and  $n = 2$ , we will briefly indicate how to recover primitive quartic extensions  $E$  of dyadic fields  $K$  which were studied by Weil [18]. We will say that  $E$  is an  $\mathfrak{A}_4$ -quartic (resp.  $\mathfrak{S}_4$ -quartic) if the group  $\text{Gal}(\hat{E}|K)$  of its galoisian closure  $\hat{E}$  is isomorphic to  $\mathfrak{A}_4$  (resp.  $\mathfrak{S}_4$ ).

(35) First let  $K = \mathbf{Q}_2$ . In view of (4), we should look for  $G_2$ -stable irreducible  $\mathbf{F}_2$ -planes in  $\overline{L_2^\times} = L_2^\times/L_2^{\times 2}$ . (Recall that  $G_2 = \text{Gal}(L_2|K)$  and  $L_2$  is the maximal abelian extension of exponent  $e_2 = 3$  of the unramified quadratic extension  $K_2 = K(\sqrt[3]{1})$  of  $K$ ).

But first let us classify degree-2  $\mathbf{F}_2$ -representations of  $G_2$ . Let  $l_2$  be the residue field of  $L_2$ , let  $T \subset G_2$  be the inertia subgroup, and let  $\theta : T \rightarrow l_2^\times$  be the canonical character. Then, in the notation of [4, 19], we have

(36) *The only irreducible degree-2  $\mathbf{F}_2$ -representations of  $G_2$  are  $\pi_{\overline{1}, \sqrt[3]{1}}$  and  $\pi_{\overline{\theta}, 1}$ .*  $\square$

(37) Concretely,  $L_2$  contains the unramified cubic extension  $L = K(\sqrt[3]{1})$  of  $K$ , and  $\pi_{\overline{1}, \sqrt[3]{1}}$  is the unique irreducible degree-2  $\mathbf{F}_2$ -representations of

$\text{Gal}(L|K) = \mathbf{Z}/3\mathbf{Z}$ . Similarly,  $L_2$  contains the unique [6, 8.1]  $\mathfrak{S}_3$ -extension  $L' = K(\sqrt[3]{1}, \sqrt[3]{2})$  of  $K$ , and  $\pi_{\overline{\theta}, 1}$  is the unique irreducible degree-2  $\mathbf{F}_2$ -representations of  $\text{Gal}(L'|K) = \mathfrak{S}_3$ . So it suffices to work separately over  $L$  and  $L'$  instead of  $L_2 = LL'$ .

(38) First take  $L = K(\sqrt[7]{1})$  and let  $G = \text{Gal}(L|K)$ . It follows from [5, 21] that the  $\mathbf{F}_2[G]$ -module  $\bar{U}_L^1$  is isomorphic to  ${}_2L^\times \oplus \mathbf{F}_2[G]$ , which contains a unique  $G$ -stable  $\mathbf{F}_2$ -plane  $D$ . By (9), there is a unique primitive quartic extension  $E$  of  $K$  whose galoisian closure is  $\hat{E} = L(\sqrt[2]{D})$ ; the group  $\text{Gal}(\hat{E}|K)$  is isomorphic to  $\mathfrak{A}_4$ , and  $\hat{E}$  is the unique  $\mathfrak{A}_4$ -extension of  $K$ .

(39) Now let  $L' = K(\sqrt[3]{1}, \sqrt[3]{2})$  and  $G = \text{Gal}(L'|K)$ . It follows as before that the  $\mathbf{F}_2[G]$ -module  $\bar{U}_{L'}^1$  is isomorphic to  ${}_2L'^\times \oplus \mathbf{F}_2[G]$ , and a finer analysis as in [5, 11] shows that there is a unique  $G$ -stable irreducible  $\mathbf{F}_2$ -plane  $D \subset \bar{U}_{L'}^5$  (such that  $D \cap \bar{U}_{L'}^6 = \{1\}$ ), and two  $G$ -stable irreducible  $\mathbf{F}_2$ -planes  $D \subset \bar{U}_{L'}^1$ , such that  $D \cap \bar{U}_{L'}^2 = \{1\}$ . Corresponding to each  $D$ , we get a primitive quartic extensions  $E$  of  $K$  whose galoisian closure is  $\hat{E} = L'(\sqrt[2]{D})$ ; the group  $\text{Gal}(\hat{E}|K)$  is isomorphic to  $\mathfrak{S}_4$  in each case, and these three are the only  $\mathfrak{S}_4$ -extensions of  $K$ .

(40) The differential exponents of these extensions can be computed as in (33). We leave for the readers (or their indefatigable computers) the pleasure of doing so and of finding equations defining them. See [11, p. 111] for the details.

(41) Similar computations can be made for  $K = \mathbf{F}_2((\varpi))$ , by working over the unramified cubic extension  $K(\sqrt[3]{1})$  (resp. the unique  $\mathfrak{S}_3$ -extension  $K(\sqrt[3]{1}, \sqrt[3]{\varpi})$ ). Now there are infinitely many  $\mathfrak{A}_4$ -quartic (resp.  $\mathfrak{S}_4$ -quartic) extensions, but only finitely many with bounded differential exponent. The  $\mathfrak{A}_4$ -quartics have to be counted carefully (as in [4, 25]), because the corresponding  $\mathbf{F}_2$ -representation is not absolutely irreducible : over the quadratic extension  $\mathbf{F}_2(\sqrt[3]{1})$  of  $\mathbf{F}_2$ , it splits into the direct sum of the two cubic characters  $\text{Gal}(K(\sqrt[3]{1})|K) \rightarrow \mathbf{F}_2(\sqrt[3]{1})^\times$ . These two characters are interchanged by the generator of the group  $\text{Gal}(\mathbf{F}_2(\sqrt[3]{1})|\mathbf{F}_2)$  of order 2.

(42) Finally allow  $K$  to be any 2-field and let  $q$  be the cardinal of its residue field. If  $q \equiv -1 \pmod{3}$ , the theory is completely similar to the cases  $q = 2$  discussed above. If  $q \equiv 1 \pmod{3}$  (so that  $K$  contains  $\sqrt[3]{1}$ ), then the group  $G_2$  is commutative of exponent 3 (and order  $3^2$ ). So apart from the unramified cubic extension, we have to deal with the three ramified cubic extensions (all three cyclic) of  $K$ . Each gives a certain (finite) number of  $\mathfrak{A}_4$ -quartic extensions of  $K$  of bounded differential exponent. There are no  $\mathfrak{S}_4$ -quartic extensions because  $K$  has no  $\mathfrak{S}_3$ -extensions.

(43) Taking  $K = \mathbf{Q}_2$  and  $n = 3$ , one can recover the list of primitive

octic extensions of  $\mathbf{Q}_2$  and their differential exponents to be found in [14]. We have  $e_3 = 2^3 - 1 = 7$ ,  $K_3 = K(\sqrt[7]{1})$ ,  $L_3 = K_2(\sqrt[7]{\xi}, \sqrt[7]{2})$  (where  $\xi$  is a generator of the multiplicative group  $k_3^\times$  of the residue field  $k_3$  of  $K_3$ ), and  $G_3 = \text{Gal}(L_3|K)$ . One has to determine the irreducible degree-3  $\mathbf{F}_2$ -representations  $\pi$  of  $G_3$ , the copies of each  $\pi$  in  $L_3^\times/L_3^{\times 2}$ , and the level (30) of each copy. The copies correspond to primitive octic extensions of  $K$  by (4) and the levels are related to their differential exponents by (33). The same computation works for  $\mathbf{F}_2((\varpi))$  upon replacing  $L_3^\times/L_3^{\times 2}$  with  $L_3^+/\wp(L_3^+)$ . We omit the details.

## 9. Historical note

(44) Let us finish by listing a few papers related to the theme of primitivity not already mentioned in the Introduction. This account is far from being a history of the subject. The concept of primitivity for subgroups of the symmetric group goes back to the *Second mémoire* (1830) of Galois ; it was clarified by Jordan in his thesis (1860) and in his *Traité* (1870).

(45) In a long series of papers beginning with [13] and culminating in [15], and in several notes in the *Compte rendus*, Krasner appears to have been the first to study wildly primitive extensions, initially over a finite extension  $K$  of  $\mathbf{Q}_p$  and later also over  $p$ -fields of characteristic  $p$ . He introduces the notion of hypergroups and extends ramification theory to finite extensions  $F$  of  $K$  which are not assumed to be galoisian over  $K$ , and, according to Arf's review of [13] in the *Zentralblatt* 18 (p. 202), gives a necessary and sufficient condition for  $F$  to be primitive. To illustrate his theory, he computes Eisenstein polynomials defining the sixteen primitive octic extensions  $E$  of  $\mathbf{Q}_2$  [14], along with information from which the differential exponent of  $E$  can be deduced. His method was taken up and generalised by Diarra [9] to  $p$ -fields of characteristic  $p$ . I haven't succeeded in penetrating their work.

(46) We have seen (28) that for a wildly primitive extension  $E$  of a  $p$ -field  $K$  with galoisian closure  $\hat{E}$ , the wild ramification subgroup of  $\text{Gal}(\hat{E}|K)$  has a unique ramification break ; in particular, it is commutative of exponent  $p$ . In [10], Fontaine studies Eisenstein polynomials over a  $p$ -field which define an abelian extension of exponent  $p$  with a unique ramification break.

(47) In [18], Weil studies  $\mathfrak{A}_4$ - and  $\mathfrak{S}_4$ -extensions of dyadic fields  $K$ . As we have seen, they are the same as galoisian closures of primitive quartic extensions, so in principle their enumeration follows from the work of Krasner and Diarra. Weil's results can be viewed as the case  $p = 2$ ,  $n = 2$  of the foregoing.

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